Int. J. Heat Mass Transfer. Vol. 36, No. 14, pp. 3647–3657, 1993
Printed in Great Britain

0017–9310/93 $6.00+0.00
Pergamon Press Ltd

Coupled mass and heat transfer in a multicomponent turbulent falling liquid film

L. P. KHOLPANOV and E. YA. KENIG
Institute of New Chemical Problems, Russian Academy of Sciences, Chernogolovka,
Moscow Region, 142432, Russia

(*Received* 1 *April* 1992)

Abstract—A method is proposed for calculating parameters of combined heat and mass transfer in a multicomponent turbulent liquid film, which is based on solving a coupled system of differential equations of convective heat conduction and of multicomponent convective diffusion. The diffusion equation and appropriate boundary conditions are of matrix character. As an example, consideration is given to the process of simultaneous absorption of two gases by a nonvolatile turbulent falling film of absorbent.

1. INTRODUCTION

THE DEVELOPMENT of a theory and methods of predicting the processes of combined heat and mass transfer arouse great interest owing to their widespread use in up-to-date industrial practice. These processes are notable for a complex structure and necessitate a simultaneous solution of momentum, substance and energy transfer equations, including coupling conditions at the interface.

Investigations of combined heat and mass transfer in laminar and wavy liquid film [1], in a liquid film flowing along a surface with regular roughness [2], in the systems consisting of a set of droplets or bubbles [3] enabled the establishment of the general functional relationship for heat transfer (mass transfer) coefficients [1]. This relationship is represented as a product, with one term being equal to the heat transfer (mass transfer) coefficient in the absence of mutual influence of heat and mass and the other being common for all types of the processes considered.

Modelling the processes of combined mass and heat transfer in multicomponent systems becomes far more complicated. Superposed diffusional interactions of the mixture components may give rise to such phenomena as reverse mass transfer, mass-transfer barrier and osmotic mass transfer [4]. All the above phenomena are characterized by the absence of a direct relation between the component driving force and mass flux (for example, in reverse mass transfer, the component driving force and its flow are directed oppositely). Therefore, they qualitatively distinguish the multicomponent processes from similar binary ones and create significant difficulties in their theoretical description.

Working out adequate methods to study and calculate the heat and mass transfer processes in moving systems is based on the employment of an apparatus of differential equations of convective mass and energy transport [5]. When multicomponent mixtures are described, the mass transfer equations are of matrix–vector form and coupled, which makes their solution difficult to accomplish. At the same time, it is exactly the matrix character of the equations, in accordance with postulates of the linear thermodynamics of irreversible processes [6], which allows the mutual influence of various transfer phenomena to be taken into account.

The current study is devoted to theoretically investigating simultaneous heat and mass transfer in a multicomponent turbulent liquid film based on a solution of the differential equations of convective heat and mass transfer. A need for taking into consideration the regularities of turbulent mass and energy transfer introduces an additional difficulty in the process analysis.

2. EQUATIONS AND PARAMETERS OF THE PROCESS MODEL

Let a liquid film fall by gravity down a vertical wall. The axis x is taken to be directed downwards along the wall and the axis y perpendicularly to it. It is assumed, as usual [7–9], that physical properties of the liquid phase are constant, the net diffusional enthalpy transfer throughout the phase volume is negligible, there are no external forces, the phase equilibrium conditions at the film interface are fulfilled and the film thickness remains unchanged. It is also presumed that a dissipation term in the heat transfer equation and substance transfer by thermal diffusion can be ignored [10]. Then the equations of steady convective heat and mass transfer in the turbulent multicomponent liquid film have the form

$$u(y)\frac{\partial T}{\partial x} = \frac{\partial}{\partial y}\left(\sigma(y)\frac{\partial T}{\partial y}\right) \qquad (1)$$

$$u(y)\frac{\partial \mathbf{C}}{\partial x} = \frac{\partial}{\partial y}\left([E(y)]\frac{\partial \mathbf{C}}{\partial y}\right) \qquad (2)$$

NOMENCLATURE

$\mathbf{C}$	vector of molar concentrations of liquid mixture components
$\mathbf{C}'$	vector of concentrations transformed using equation (17)
$\mathbf{c}$	vector of dimensionless concentrations obtained using equation (20)
$[D]$	matrix of multicomponent diffusion coefficients
$\lceil D' \rfloor$	diagonal matrix consisting of eigenvalues of matrix $[D]$
$[E]$	matrix of multicomponent turbulent diffusion coefficients, equation (10)
$[\bar{E}]$	film thickness average matrix $[E]$, $[D]+\bar{\varepsilon}\lceil I \rfloor$
$\lceil E' \rfloor$	diagonal matrix consisting of eigenvalues of matrix $[E]$
$\mathbf{e}$	unit vector
Fr	Froude number, $\bar{u}^2/h_0 g$
g	acceleration due to gravity
ΔH_i	molar enthalpy difference of component i in gaseous and liquid phases transferred by its mass flux across the interface
h_0	liquid film thickness
$\lceil I \rfloor$	unit matrix
$[L]$	fundamental matrix for $[D]$ and $[E]$
$\mathbf{N}$	vector of molar fluxes of mixture components
n	number of mixture components
Pr	Prandtl number, ν/κ
Pr	film thickness average turbulent Prandtl number, $\nu/\bar{\sigma}$
p	parameter of analytical solution defined by equation (62)
q	heat flux
Re	Reynolds number, $4\bar{u}h_0/\nu$
Sc'_i	modified Schmidt number of the component i, ν/D'_i
$\overline{Sc}'_i$	film thickness average turbulent modified Schmidt number of the component i, ν/E'_i
u	liquid film flow velocity
$\bar{u}$	integral mean over the film thickness value of u
$\mathbf{V}$	vector of parameters of the equilibrium relation (5)
v	dimensionless film velocity defined by equation (19)
$\mathbf{W}$	vector of parameters of the equilibrium relation (5)
x	coordinate along the film flow
y	coordinate across the film flow.

Greek symbols

ε	turbulent constituent of transfer coefficients
$\bar{\varepsilon}$	film thickness average value of ε
η	dimensionless coordinate defined by equation (19)
η_1	$1-\eta$
κ	thermal diffusivity
λ	thermal conductivity
ν	kinematic viscosity
ξ	dimensionless coordinate defined by equation (19)
σ	eddy thermal diffusivity
$\bar{\sigma}$	film thickness average value of σ, $\kappa+\bar{\varepsilon}$.

Subscripts

0	contact device entrance
e	equilibrium
i	component i
w	contact device wall.

Superscripts

$'$	quantity transformed using the fundamental matrix
(T)	row-vector
$\bar{\ }$	averaged value.

where $\sigma(y)$ is the eddy thermal diffusivity and $[E(y)]$ is the matrix of turbulent diffusion coefficients. These quantities are dependent both on physicochemical properties of the liquid mixture and on hydrodynamic parameters. The column-vector $\mathbf{C}$ consists of molar concentrations of the components C_i and has the dimension $n-1$ [11].

Boundary conditions are prescribed by the following relations:

at the entrance

$$x = 0, \quad T = T_0, \quad \mathbf{C} = \mathbf{C}_0 \tag{3}$$

at the wall (the adiabaticity and impermeability conditions)

$$y = 0, \quad \frac{\partial T}{\partial y} = 0, \quad \frac{\partial C_i}{\partial y} = 0, \quad i = 1, 2, \ldots, n-1 \tag{4}$$

at the film interface (the phase equilibrium and heat balance conditions)

$$y = h_0, \quad \mathbf{C} = \mathbf{V}T + \mathbf{W}, \tag{5}$$

$$q = \sum_{k=1}^{n} N_k \Delta H_k. \tag{6}$$

To express the heat and mass fluxes in terms of gradients of temperature and component concentrations, use must be made of the Fourier law and the generalized Fick law. Furthermore, an additional condition of mass flux relation, which has different forms for different processes, is imposed on the system.

For instance, for the process of multicomponent distillation, with the equimolarity condition ($\Sigma_{k-1}^{n} N_k = 0$) used, relation (6) acquires the form

$$\lambda \frac{\partial T}{\partial y} = \mathbf{r}_p^{(\mathrm{T})}[D]\frac{\partial \mathbf{C}}{\partial y} \tag{7}$$

where $r_{pi} = \Delta H_i - \Delta H_n$; $i = 1, 2, \ldots, n-1$, whereas, for multicomponent absorption with the inert component ($N_n = 0$), relation (6) looks as follows:

$$\lambda \frac{\partial T}{\partial y} = \mathbf{r}_{\mathrm{A}}^{(\mathrm{T})}[D]\frac{\partial \mathbf{C}}{\partial y} \tag{8}$$

where $r_{\mathrm{A}i} = \Delta H_i$; $i = 1, 2, \ldots, n-1$. Therefore, condition (6) in the general case is written in the form

$$\lambda \frac{\partial T}{\partial y} = \mathbf{r}^{(\mathrm{T})}[D]\frac{\partial \mathbf{C}}{\partial y} \tag{9}$$

where $r = r_{pi}$ for the multicomponent distillation and $r = r_{\mathrm{A}i}$ for the multicomponent absorption with the inert component, etc.

The matrix of the turbulent diffusion coefficients entering into equation (2) is of the following structure [12–15]:

$$[E(y)] = [D] + \varepsilon_D(y)\lceil I \rfloor \tag{10}$$

where ε_D is the turbulent diffusion coefficient which is specified by hydrodynamic conditions and can be found from the results of studying turbulence in binary mixtures [15].

The matrix $[D]$ is expressed in terms of the molecular diffusion coefficients of binary pairs of the mixture components. Equations for calculating $[D]$ can be obtained with the aid of a molecular theory of fluids [10, 11, 16]. Salient features of the matrix $[D]$ are reducibility to a diagonal form as well as realness and positiveness of the eigenvalues [7, 16].

The matrix $[E]$ defined by equation (10) will be shown to possess the same properties.

Indeed, let $[L]$ be the fundamental matrix for $[D]$, i.e. $[L]^{-1}[D][L] = \lceil D' \rfloor$.

Then

$$\begin{aligned}[L]^{-1}[E(y)][L] &= [L]^{-1}[D][L] + [L]^{-1}\varepsilon_D(y)\lceil I \rfloor [L] \\ &= \lceil D' \rfloor + \varepsilon_D(y)\lceil I \rfloor = \lceil E' \rfloor\end{aligned}$$

where $E'_i = D'_i + \varepsilon_D$, $i = 1, 2, \ldots, n-1$.

Since $D'_i > 0$ and $\varepsilon_D > 0$, then $E'_i > 0$ for all i. Consequently, the matrix $[E(y)]$ is reducible to the diagonal form (with the help of the same fundamental matrix as that for $[D]$) and has real positive eigenvalues.

The eddy thermal diffusivity coefficient entering into equation (1), similarly to the matrix $[E(y)]$, comprises two terms [8]:

$$\sigma(y) = \kappa + \varepsilon_{\mathrm{T}}(y) \tag{11}$$

where $\varepsilon_{\mathrm{T}}(y)$ is the thermal analogue of the turbulent diffusion coefficient $\varepsilon_D(y)$.

3. TRANSFORMATION OF THE SYSTEM OF EQUATIONS

The system of equations (1)–(2) with the boundary conditions (3)–(5), and (9) is a coupled system of parabolic equations with 4th-kind matrix coupled boundary conditions, which sets up appreciable difficulties in its solution.

Therefore, use will be made of the diagonalizability property of the matrices $[D]$ and $[E]$ and, applying the method proposed in ref. [17], the initial system will be transformed to the uncoupled form

$$u(y)\frac{\partial T}{\partial x} = \frac{\partial}{\partial y}\left(\sigma \frac{\partial T}{\partial y}\right) \tag{12}$$

$$u(y)\frac{\partial \mathbf{C}'}{\partial x} = \frac{\partial}{\partial y}\left(\lceil E' \rfloor \frac{\partial \mathbf{C}'}{\partial y}\right). \tag{13}$$

The boundary conditions are:

for

$$x = 0, \quad T = T_0, \quad \mathbf{C}' = \mathbf{C}'_0 \tag{14}$$

for

$$y = 0, \quad \frac{\partial T}{\partial y} = 0, \tag{15a}$$

$$\frac{\partial \mathbf{C}'}{\partial y} = 0 \tag{15b}$$

for

$$y = h_0, \quad \mathbf{C}' = \mathbf{V}'T + \mathbf{W}', \quad \lambda \frac{\partial T}{\partial y} = \mathbf{r}_1^{(\mathrm{T})}\frac{\partial \mathbf{C}'}{\partial y} \tag{16}$$

where

$$\mathbf{C}' = [L]^{-1}\mathbf{C}, \quad \mathbf{C}'_0 = [L]^{-1}\mathbf{C}_0 \tag{17}$$

$$\mathbf{V}' = [L]^{-1}\mathbf{V}, \quad \mathbf{W}' = [L]^{-1}\mathbf{W}, \quad \mathbf{r}_1^{(\mathrm{T})} = \mathbf{r}^{(\mathrm{T})}[D][L]. \tag{18}$$

Introduce dimensionless variables via the ratios

$$\xi = \frac{4x}{Re\, h_0}, \quad \eta = \frac{y}{h_0}, \quad v = \frac{u}{\bar{u}}, \quad t = \frac{T - T_0}{T_{\mathrm{e}} - T_0} \tag{19}$$

$$c_i = \frac{C'_i - \mathbf{C}'_{0i}}{C'_{\mathrm{e}i} - C'_{0i}}, \quad i = 1, 2, \ldots, n-1 \tag{20}$$

where

$$C'_{\mathrm{e}i} = V'_i T_0 + W'_i, \quad T_{\mathrm{e}} = (C_{01} - W_1)/V_1. \tag{21}$$

Then the system (12)–(16) takes the final form

$$v(\eta)\frac{\partial t}{\partial \xi} = \frac{\partial}{\partial \eta}\left\{\left(Pr^{-1}+\frac{\varepsilon_T}{v}\right)\frac{\partial t}{\partial \eta}\right\} \tag{22}$$

$$v(\eta)\frac{\partial \mathbf{c}}{\partial \xi} = \frac{\partial}{\partial \eta}\left\{\left(\lceil Sc'\rfloor^{-1}+\frac{\varepsilon_D}{v}\lceil I\rfloor\right)\frac{\partial \mathbf{c}}{\partial \eta}\right\} \tag{23}$$

$$\xi = 0, \quad t = 0, \quad c_i = 0 \tag{24}$$

$$\eta = 0, \quad \frac{\partial t}{\partial \eta} = 0, \quad \frac{\partial c_i}{\partial \eta} = 0 \tag{25}$$

$$\eta = 1, \quad c_i + a_i t = 1, \tag{26}$$

$$\frac{\partial t}{\partial \eta} = \sum_{k=1}^{n-1} b_k \frac{\partial c_k}{\partial \eta}, \quad i = 1, 2, \ldots, n-1 \tag{27}$$

where

$$a_i = v_i' \frac{T_0 - T_c}{C'_{ci} - C'_{0i}}, \quad b_i = \frac{r_{1i}(C'_{ci} - C'_{0i})}{\lambda(T_c - T_0)}, \tag{28}$$

$$Sc_i' = \frac{v}{D_i'}. \tag{29}$$

4. DETERMINATION OF THE TURBULENT TRANSFER COEFFICIENTS

Let us assume that the coefficients of turbulent diffusion and eddy thermal diffusivity are equal [18]

$$\varepsilon_D(\eta) = \varepsilon_T(\eta) = \varepsilon(\eta). \tag{30}$$

Relation (30) is rather a complex function, since the character of variation in the turbulent transfer coefficient near the wall, in the flow core and at the phase interface is different [19]. There are extensive investigations concerned with determining the functional relation (30) for various layers of the film flow [8, 19–21]. Study [18] contains a thorough analysis of those investigations, and suggests using the following three expressions.

For the near-wall region, the van Driest equation [22] is recommended

$$\frac{\varepsilon}{v} = \frac{1}{2}\left\{-1+\left[1+0.04\frac{Re^2\eta^2}{Fr}\right.\right.$$
$$\left.\left.\times\left(1-\exp\left(-\frac{Re\,\eta}{104\sqrt{(Fr)}}\right)\right)^2\right]^{1/2}\right\} \tag{31}$$

where

$$0 \leqslant \eta \leqslant \eta_1^*, \quad \eta_1^* = 120\sqrt{(Fr)}/Re. \tag{32}$$

For the turbulent flow core, the Reichardt equation [23] is advised

$$\frac{\varepsilon}{v} = K\frac{Re}{\sqrt{(Fr)}}(2\eta-\eta^2)(3-4\eta+2\eta^2) \tag{33}$$

where the constant K is obtained from the condition that the results of computation by equations (31) and (33) coincide at the point $\eta = \eta_1^*$ determined from equation (32).

For the boundary region, a modification of the Lamourelle and Sandall equation [24] is recommended

$$\frac{\varepsilon}{v} = 3.153\times 10^{-7}\,Fr^{-2/3}\,Re^{2.758}(1-\eta)^2 \tag{34}$$

which is utilized for the dimensionless thickness ranging from $\eta = 1$ to the value $\eta = \eta_2^*$, at which the results of calculations by equations (33) and (34) are in agreement.

To determine the profile of turbulent diffusion coefficient from equations (31)–(34), two computational procedures need to be constructed. The first of them is intended for calculating the constant K in equation (33) by the known Re and Fr values and at $\eta = \eta_1^*$. The second procedure determines the dimensionless thickness η_2^* from the familiar Re and Fr values, and from the K value calculated in the first procedure. Both the procedures are realized in the present study using the Newton method.

Figure 1 plots the characteristic variation in the turbulent diffusion coefficient across the film thickness obtained from equations (31)–(34) for various Re and Fr values. The plots clearly indicate the region of near-wall film flow, the turbulent flow core and the

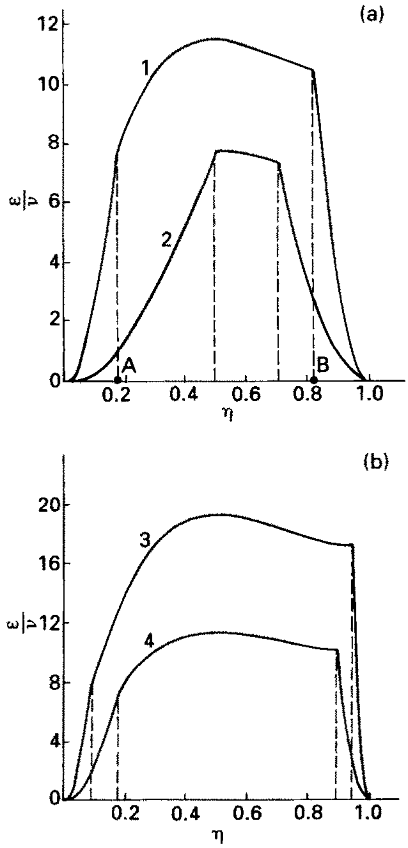

FIG. 1. Dimensionless turbulent diffusion coefficient vs dimensionless film thickness for different Re and Fr: (1) $Re = 5000$, $Fr = 54.5$; (2) $Re = 5000$, $Fr = 436$; (3) $Re = 20\,000$, $Fr = 218$; (4) $Re = 10\,000$, $Fr = 218$.

boundary region (they are separated by a dot line). For curve 1, in particular, those regions conform to the intervals 0–A, A–B and B–1.0.

5. DETERMINATION OF THE VELOCITY PROFILE

To obtain the equation defining the velocity distribution across the film thickness, the steady film flow equation [8] is employed in the form

$$\frac{\mathrm{d}}{\mathrm{d}y}\left\{\left(\nu+\varepsilon(y)\right)\frac{\mathrm{d}u}{\mathrm{d}y}\right\}+g=0.$$

Its integration with allowance for the boundary condition at the interface $y = h_0$ and $\mathrm{d}u/\mathrm{d}y = 0$, yields

$$(\nu+\varepsilon(y))\frac{\mathrm{d}u}{\mathrm{d}y}=g(h_0-y). \qquad (35)$$

Equation (35) is then integrated taking account of the boundary condition at the wall ($y = 0$ and $u = 0$):

$$u(y)=g\int_0^y\frac{h_0-y}{\nu+\varepsilon(y)}\,\mathrm{d}y,$$

or, in the dimensionless form (see equation (19)):

$$v(\eta)=\frac{gh_0^2}{\bar{u}\nu}\int_0^\eta\frac{1-\eta}{1+\dfrac{\varepsilon(\eta)}{\nu}}\,\mathrm{d}\eta$$

$$=\frac{Re}{4\,Fr}\int_0^\eta\frac{1-\eta}{1+\dfrac{\varepsilon(\eta)}{\nu}}\,\mathrm{d}\eta. \qquad (36)$$

Using the definition of the integral mean flow velocity, written in the dimensionless form

$$\int_0^1 v(\eta)\,\mathrm{d}\eta=1, \qquad (37)$$

a criterion is obtained which enables the parameter Fr to be found from the assigned Re [18]. To determine Fr, the current study again employs the Newton method utilizing equations (31), (33), (34) and (36).

Thus, the solution for the dimensionless problem (22)–(27) in view of equations (30)–(34) and (36)–(37) is governed by the dimensionless parameters Re, Pr, Sc_i', a_i, b_i, $i = 1, 2, \ldots, n-1$, i.e. is dependent on the $3n-1$ quantities.

6. SOLUTION

To calculate the fundamental matrices $[L]$, the method suggested in ref. [25] is applied.

The transformed system (22)–(27) is solved by the factorization method using finite-difference operators [26]:

$$\frac{\partial W}{\partial\xi}=\frac{W(k,j)-W(k-1,j)}{\Delta\xi},$$

$$\frac{\partial}{\partial\eta}\left\{Q(\eta)\frac{\partial W}{\partial\eta}\right\}=\frac{1}{(\Delta\eta)^2}$$

$$\times\left\{\frac{Q(j+1)+Q(j)}{2}[W(k,j+1)-W(k,j)]\right.$$

$$\left.-\frac{Q(j)+Q(j-1)}{2}[W(k,j)-W(k,j-1)]\right\} \qquad (38)$$

where k and j are the numbers of steps along the vertical and horizontal axes, $\Delta\xi$ and $\Delta\eta$ are the relevant magnitudes of steps, whereas W and Q represent, accordingly, either t and σ, or c_i and E_i', $i = 1, 2, \ldots, n-1$.

Since the relationships between dimensionless concentrations and temperature as well as between their derivatives at the film interface retain a coupled character (equations (26) and (27)), defining the boundary factorization coefficients requires that the system of algebraic equations should be solved at each step along the vertical coordinate. Here, for improving accuracy of the derivative approximation at the film interface, use is made of finite-difference analogues which allow approximation with an accuracy of third-order of smallness [27]:

$$\left.\frac{\mathrm{d}W}{\partial\eta}\right|_{\eta=1}=\frac{1}{12(\Delta\eta)}\{8f(k)-18W(k,N-1)$$

$$+W(k,N-2)+9W(k,N)\} \qquad (39)$$

where N is the total number of steps along the horizontal coordinate and $f(k)$ is the value of the appropriate function at the interface defined for any k by the equation

$$f(k)=\frac{W(k,N-1)+W(k,N)}{2}. \qquad (40)$$

Having found, via the factorization method, the distributions of dimensionless concentrations of the components $c_i(\xi,\eta)$, $i = 1, 2, \ldots, n-1$ and of the temperature $t(\xi,\eta)$, it is possible to predict any characteristics of multicomponent heat and mass transfer, e.g. the values of near-wall and interface concentrations and temperatures, their integral mean values given by the equations

$$\bar{c}(\xi)=\int_0^1 v(\eta)c(\xi,\eta),\quad \bar{t}(\xi)=\int_0^1 v(\eta)t(\xi,\eta)\,\mathrm{d}\eta,$$

$$(41)$$

the mass and heat fluxes of the components at the interface, the mass and heat transfer coefficients, etc.

7. SOLUTION FOR LONG DISTANCES FROM THE ENTRANCE

Let us now consider the development of the heat and mass transfer process for long lengths of a contacting device. For $\xi\to\infty$, the component concentrations and film temperature tend to equilibrium constant values (to asymptotics).

To determine the asymptotics, use is made of the relations

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\int_0^1 v(\eta)c_i/(\xi,\eta)\,\mathrm{d}\eta = \frac{1}{Sc_i'}\left.\frac{\partial c_i}{\partial \eta}\right|_{\eta=1}$$

$$i = 1, 2, \ldots, n-1\,;$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\int_0^1 v(\eta)t(\xi,\eta)\,\mathrm{d}\eta = \frac{1}{Pr}\left.\frac{\partial t}{\partial \eta}\right|_{\eta=1} \tag{42}$$

which can be obtained by integrating the convective diffusion and heat conduction equations over the film thickness with consideration of the continuity equation.

The boundary condition (27) yields

$$\frac{1}{Pr}\left.\frac{\partial t}{\partial \eta}\right|_{\eta=1} = \frac{1}{Pr}\sum_{k=1}^{n-1} b_k \left.\frac{\partial c_k}{\partial \eta}\right|_{\eta=1} = \sum_{k=1}^{n-1}\omega_k \frac{1}{Sc_k'}\left.\frac{\partial c_k}{\partial \eta}\right|_{\eta=1} \tag{43}$$

where

$$\omega_k = \frac{b_k\, Sc_k'}{Pr}.$$

In view of equation (43), equation (42) gives

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\int_0^1 v(\eta)t(\xi,\eta)\,\mathrm{d}\eta = \sum_{k=1}^{n-1}\omega_k\frac{\mathrm{d}}{\mathrm{d}\xi}\int_0^1 v(\eta)c_k(\xi,\eta)\,\mathrm{d}\eta$$

or (see equation (41))

$$\frac{\mathrm{d}}{\mathrm{d}\xi}(\bar{t}-\boldsymbol{\omega}^{(\mathrm{T})}\bar{\mathbf{c}}) = 0. \tag{44}$$

It follows from equation (44) that the expression in brackets is the same for any ξ. Particularly, for $\xi = 0$, when the conditions (24) are fulfilled, this expression is equal to zero, and hence

$$\bar{t} = \boldsymbol{\omega}^{(\mathrm{T})}\bar{\mathbf{c}}. \tag{45}$$

Equality (45) is a dimensionless form of the heat balance for a contacting device section of arbitrary length, starting from $\xi = 0$.

Now let us taken into account that, for $\xi \to \infty$,

$$\bar{\mathbf{c}} = \mathbf{c}|_{\eta=1}, \quad \bar{t} = t|_{\eta=1}$$

and make use of the boundary condition (26):

$$\bar{\mathbf{c}}+\mathbf{a}\bar{t} = \mathbf{e}. \tag{46}$$

Substituting equation (45) into equation (46) results in

$$\bar{\mathbf{c}}+\mathbf{a}(\boldsymbol{\omega}^{(\mathrm{T})}\bar{\mathbf{c}}) = \mathbf{e}. \tag{47}$$

To solve the vector equation (47) with respect to $\bar{\mathbf{c}}$, it is written for the i-component as

$$\bar{c}_i + a_i\sum_{k=1}^{n-1}\omega_k\bar{c}_k = 1$$

and transformed:

$$\bar{c}_i(1+a_i\omega_i)+a_i\sum_{k\neq i}^{n-1}\omega_k\bar{c}_k = 1.$$

The matrix $[\Psi]$ is defined as

$$[\Psi]_{ii} = 1+a_i\omega_i, \quad [\Psi]_{ik} = a_i\omega_k \quad (i \neq k). \tag{48}$$

Then, instead of equation (47), we obtain

$$[\Psi]\bar{\mathbf{c}} = \mathbf{e}. \tag{49}$$

The asymptotics can then be found from equations (45) and (49):

$$\bar{\mathbf{c}}(\xi \to \infty) = [\Psi]^{-1}\mathbf{e},$$

$$\bar{t}(\xi \to \infty) = \boldsymbol{\omega}^{(\mathrm{T})}[\Psi]^{-1}\mathbf{e}. \tag{50}$$

8. NUMERICAL EXAMPLE AND DISCUSSION OF RESULTS

By way of example, the current study has numerically investigated the process of three-component nonisothermal absorption in which a two-component gas mixture is absorbed by a nonvolatile turbulent liquid film. Figures 2–7 give the characteristic depen-

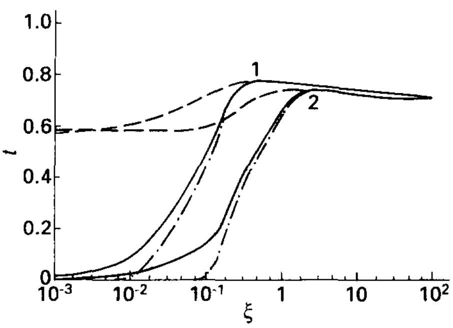

FIG. 2. Dimensionless temperature vs the contacting device length at $Pr = 10$, $Sc_1' = 1500$, $Sc_2' = 2500$, $a_1 = 1.1$, $a_2 = 0.9$, $b_1 = 0.3$, $b_2 = -0.1$ for different Re: (1) $Re = 2000$; (2) $Re = 5000$ (solid lines conform to values in the phase core, dash lines to values at the film interface and dash-and-dot lines to values at the wall).

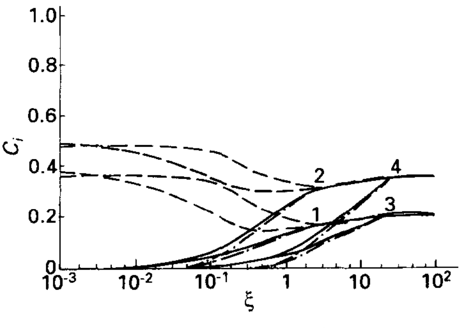

FIG. 3. Dimensionless concentrations of components vs the contacting device length (the values of parameters and notation are the same as in Fig. 2); (1), (2) $Re = 20\,000$ ((1) c_1; (2) c_2); (3), (4) $Re = 5000$ ((3) c_1, (4) c_2).

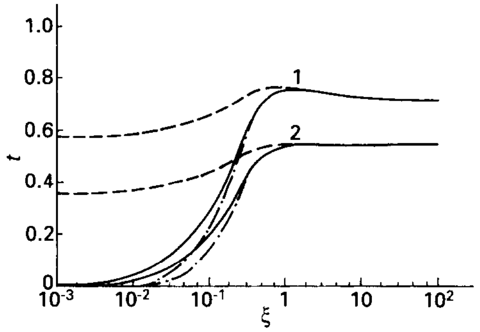

FIG. 4. Dimensionless temperature vs the contacting device length at $Re = 10000$, $Pr = 10$, $a_1 = 1.1$, $a_2 = 0.9$, $b_1 = 0.3$, $b_2 = -0.1$ for different Sc'_i (notation is the same as in Fig. 2): (1) $Sc'_1 = 1500$, $Sc'_2 = 2500$; (2) $Sc'_1 = 150$, $Sc'_2 = 250$.

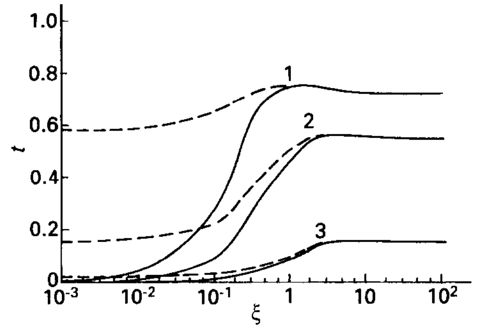

FIG. 6. Dimensionless temperature vs the contacting device length at $Re = 10000$, $Pr = 10$, $Sc'_1 = 1500$, $Sc'_2 = 2500$, $a_1 = 1.1$, $a_2 = 0.9$ for different b_i: (1) $b_1 = 0.3$, $b_2 = -0.1$; (2) $b_1 = 0.03$, $b_2 = -0.01$; (3) $b_1 = 0.003$, $b_2 = -0.001$ (solid lines conform to values in the phase core, dashed lines to values at the film interface).

dences of dimensionless concentrations of the components and temperatures on the contacting device length for various governing parameters of the problem. Solid lines show the temperature and concentration profiles in the phase core (integral mean over the film thickness values, equation (41)), dash lines denote the profiles of interface values and dot-and-dash lines represent the profiles of near-wall values.

Clearly, on the initial section of the contacting device, the temperature and concentrations drops across the film thickness are maximal and, therefore, the motive forces of heat and mass transfer, namely the gradients of temperature and concentrations of the components, are maximal as well. With increasing ξ, the transfer rate starts decreasing, the motive forces reduce and, ultimately, the process ends up by going to the equilibrium temperature and concentrations values. Naturally, in this case the relevant values at the wall, in the flow core and at the film interface become identical and equal, respectively, to $\bar{t}(\xi \to \infty)$ and $\bar{c}(\xi \to \infty)$ (equations (50)). This allows the effective length of the apparatus to be readily identified.

By using equations (19)–(21), conversion is effected from the dimensionless parameters of the problem to those dimensional and, thereafter, with the help of equations (17), to the real concentration values of the components in the turbulent liquid film.

The problem considered is multiparametric, and studying the influence of various parameters on the solution can be of profound interest. The suggested prediction method enables such an analysis to be performed on the basis of the distributions obtained.

Figures 2 and 3 present the corresponding distributions for different values of Re. The higher the film flow velocity, the more intense the transfer process. Therefore, with increasing Re, the section of active heat and mass transfer (a rapid variation of temperature and concentrations of the components) shifts to the left, to smaller values of the dimensionless length ξ. Inasmuch as the asymptotics (equations (50)) are unaffected by Re, curves 1 and 2 in Fig. 2, as well as 1 and 3, 2 and 4 in Fig. 3 coincide, when $\xi \to \infty$.

Figures 3 and 4 give the distributions of temperature and concentrations of the components for different values of Sc'_i. A decrease in Sc'_1 and Sc'_2, with

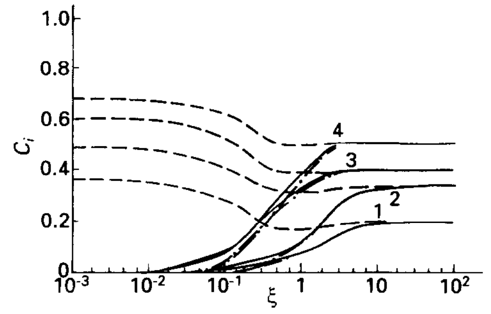

FIG. 5. Dimensionless concentrations of components vs the contacting device length (the values of parameters and notation same as in Fig. 4): (1), (2) $Sc'_1 = 1500$, $Sc'_2 = 2500$ ((1) c_1; (2) c_2); (3), (4) $Sc'_1 = 150$, $Sc'_2 = 250$ ((3) c_1; (4) c_2).

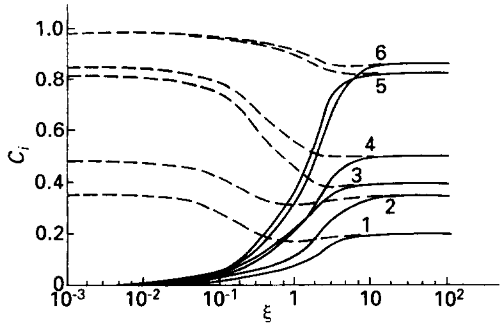

FIG. 7. Dimensionless concentrations of components vs the contacting device length (the values of parameters and notation same as in Fig. 6): (1), (2) $b_1 = 0.3$, $b_2 = -0.1$ ((1) c_1; (2) c_2); (3), (4) $b_1 = 0.03$, $b_2 = -0.01$ ((3) c_1; (4) c_2); (5), (6) $b_1 = 0.003$, $b_2 = -0.001$ ((5) c_1; (6) c_2).

other parameters unchanged, implies that the mass transfer rate increases as against the heat transfer rate. On the one hand, this leads to a shift of the temperature curves to the right (Fig. 4) and to a shift of the concentration curves to the left (Fig. 5). On the other hand, according to equations (50), the asymptotic temperature values fall (Fig. 4), whereas the asymptotic values of concentrations of the components rise (Fig. 5).

Figures 6 and 7 present the temperature and concentration profiles for different b_i. With increasing absolute values of b_1 and b_2, which are proportional to the appropriate absorption heats of the components (see equation (28)), the film heating is enhanced (Fig. 6) and concentrations of the components dissolved in it diminish (Fig. 7).

It was also interesting to study the effect of the type of the profile of the film flow velocity $v(\eta)$ on the distributions of heat and mass transfer characteristics. For this end, along with the profile (36), the current study employed a plane profile corresponding to the film flow at a flow-rate mean velocity:

$$v(\eta) = \bar{v} = 1. \tag{51}$$

The analysis revealed that the effect of the velocity profile on the distributions of temperature and concentrations of the components is very small. Deviations between the temperature curves obtained from equations (36) and (51) are not larger than 3–4%, whereas, for the concentration curves, those deviations are still smaller. Thus, the use of the plane velocity profile for predicting heat and mass transfer in the turbulent liquid film produces an inconsiderable error. This inference is consistent with the results of studying mass transfer in a laminar film flow [25], the error with the plane velocity profile as applied to the turbulent flow being still smaller than that applied to the laminar flow.

Apart from the adiabatic boundary condition (15a), the present study has also employed the 1st-kind boundary condition conforming to isothermal wall

$$y = 0, \quad T = T_w \tag{52}$$

or, in dimensionless form,

$$\eta = 0, \quad t = t_w \tag{53}$$

where

$$t_w = \frac{T_w - T_0}{T_c - T_0}.$$

Using the boundary condition (52) instead of (15a) results in an appreciable variation of the temperature and concentration distributions. Figure 8 gives the temperature dependences, and Fig. 9 the corresponding concentration dependences, which enable comparison of the results obtained under adiabatic and isothermal conditions at the wall ($t_w = 0$).

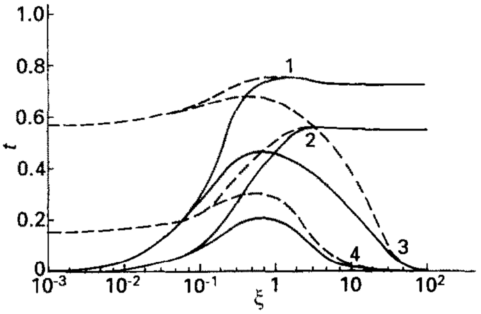

FIG. 8. Dimensionless temperature vs the contacting device length at $Re = 10\,000$, $Pr = 10$, $Sc'_c = 1500$, $Sc'_2 = 2500$, $a = 1.1$, $a_2 = 0.9$, $t_w = 0$ for different temperature boundary conditions at the wall (notation is the same as in Fig. 6) : (1), (2) condition (15a) ; (3), (4) condition (52) ; (1), (3) $b_1 = 0.3$, $b_2 = -0.1$; (2), (4) $b_1 = 0.03$, $b_2 = -0.01$.

The influence of the parameter t_w on the problem solution is illustrated in Figs. 10 and 11. An asymptotic value (when $\xi \to \infty$) for the temperature profile, with the boundary condition (53) used is equal to t_w. By substituting this quantity into condition (26), the asymptotics of the concentration curves are found:

$$\bar{\mathbf{c}}(\xi \to \infty) = \mathbf{e} - \mathbf{a}t_w. \tag{54}$$

It follows from the above equation that, at $t_w = 0$, the asymptotic values of dimensionless concentrations of all components are equal to unity. This is evident from Fig. 9 (curves 5–8) and Fig. 11 (curves 1–2).

It should be noted that the plots given in the figures and some conclusions drawn from their analysis agree with the results of study [18] dealing with a similar problem in a binary system. However, the problem considered here is much more complicated, because the studied process of coupled heat and mass transfer studied is multicomponent.

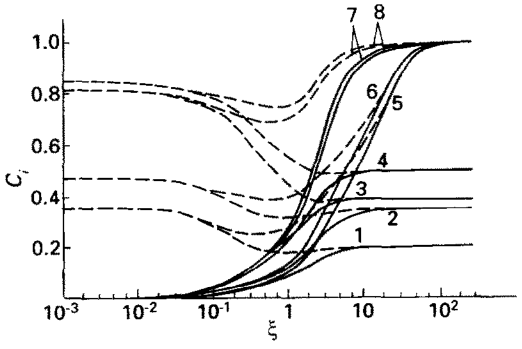

FIG. 9. Dimensionless concentrations of components vs the contacting device length (the values of parameters and notation same as in Fig. 8): (1)–(4), condition (15a); (5)–(8), condition (52); (1)–(2), (5)–(6), $b_1 = 0.3$, $b_2 = -0.1$ ((1), (5) c_1; (2), (6), c_2); (3)–(4), (7)–(8), $b_1 = 0.03$, $b_2 = -0.01$ ((3), (7) c_1; (4), (8) c_2).

Indeed, the crossing effects of multicomponent diffusion manifest themselves both near the contacting device wall and close to the film interface, i.e. where

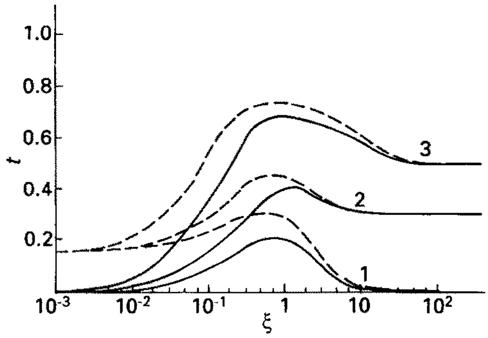

FIG. 10. Dimensionless temperature vs the contacting device length at $Re = 10000$, $Pr = 10$, $Sc'_1 = 1500$, $Sc'_2 = 2500$, $a_1 = 1.1$, $a_2 = 0.9$, $b_1 = 0.03$, $b_2 = -0.01$ for the boundary condition of an isothermal wall (53) and different t_w (notation same as in Fig. 6): (1) $t_w = 0$; (2) $t_w = 0.3$; (3) $t_w = 0.5$.

turbulent oscillations are damped and molecular diffusion begins prevailing over turbulent diffusion (see equations (10), (30)–(34)). Moreover, they are also exhibited at the film interface, in the heat balance condition (9), whereas the equilibrium relations in the multicomponent case are of vector form (5). As a consequence, instead of 4 parameters governing the binary problem solution [18], we obtain $3n-1$ of such parameters. This means that, even for the three-component example considered, the number of quantities affecting the solution is already equal to 8.

A qualitative distinction between multicomponent and binary transfer is also reflected in the plots. For example, in the binary problem [18], extrema on the temperature and concentration profiles were only observable in the isothermal wall case (condition (52)). Had the adiabatic condition (15a) at the wall be fulfilled, the concentrations and temperature distributions represented monotonic functions of the contacting device length. At the same time, as clear from Figs. 2–9 of the given study, extrema of temperature and concentration profiles is a widespread phenomenon in multicomponent heat and mass transfer, irrespective of the selection of a heat boundary condition at the wall.

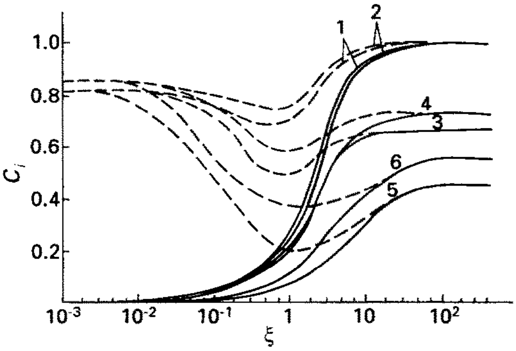

FIG. 11. Dimensionless concentration of components vs the contacting device length (the values of parameters and notation same as in Fig. 10): (1), (2) $t_w = 0$ ((1) c_1; (2) c_2); (3), (4) $t_w = 0.3$ ((3) c_1; (4) c_2); (5), (6) $t_w = 0.5$ ((5) c_1; (6) c_2).

In the case under consideration, the mutual influence of heat and mass transfer also gets complicated. Thus, for instance, the liquid film heating or cooling can bring about an alteration in the direction of mass transfer of individual mixture components. In turn, the interaction of diffusion fluxes, with regard to the film interface thermodynamics, affects the heat transfer rate.

The coupled process studied is essentially a combination of the multicomponent interface interaction of the components as well as of multicomponent diffusion and heat conduction in the liquid film, which are complicated by its gravitational flow and turbulence. Overall, the process is governed by quite a number of different, sometimes opposite, motive forces, which is precisely the fact that predetermines its intricacy.

9. SIMPLIFIED MODEL. ANALYTICAL SOLUTION

Based on numerical solving of the equations of multicomponent substance and energy transfer, it is difficult to identify an analytical relationship of the mutual influence of heat and mass transfer for binary mixtures, since the multicomponent problem is dependent on a large number of affecting parameters.

However, in this case the problem formulation is simplified and it is assumed that $[E]$, σ and u within the range under consideration can be used as average values, then the system of equations (1) and (2) with the boundary conditions (3)–(5) and (9) admits an analytical solution in the boundary-layer approximation.

Here, the wall conditions are replaced by the following:

$$y \to -\infty, \quad T = T_0, \quad \mathbf{C} = \mathbf{C}_0. \tag{55}$$

Let $\eta_1 = 1-\eta$. Then diagonalization and non-dimensionalizing carried out to the foregoing procedure lead to the problem

$$\frac{\partial t}{\partial \xi} = \frac{1}{Pr}\frac{\partial^2 t}{\partial \eta_1^2}; \quad \frac{\partial c_i}{\partial \xi} = \frac{1}{Sc'_i}\frac{\partial^2 c_i}{\partial \eta_1^2} \tag{56}$$

$$\xi = 0, \quad t = 0, \quad c_i = 0 \tag{57}$$

$$\eta_1 = 0, \quad c_i + a_i t = 1, \quad \frac{\partial t}{\partial \eta_1} = \sum_{k=1}^{n-1} b_k \frac{\partial c_k}{\partial \eta_1} \tag{58}$$

$$\eta_1 \to \infty, \quad t = 0, \quad c_i = 0. \tag{59}$$

The system (56)–(59) is solved using the analytical method proposed in refs. [17, 28], and the resulting dimensionless distributions of temperature and concentrations of the components have the form

$$t = \left[1 - \operatorname{erf}\left(\frac{\sqrt{(Pr)}}{2\sqrt{(\xi)}}\eta_1\right)\right];$$

$$\mathbf{c} = (\lceil I \rfloor - \lceil S' \rfloor)(\mathbf{e} - \mathbf{a}p),$$

where

$$p = \frac{\sum_{k=1}^{n-1} b_k \sqrt{(\overline{Sc'_k})}}{\sqrt{(\overline{Pr})} + \sum_{k=1}^{n-1} a_k b_k \sqrt{(\overline{Sc'})}}; \quad S'_i = \operatorname{erf}\left(\frac{\sqrt{(\overline{Sc'_i})}}{2\sqrt{(\xi)}}\eta_1\right).$$

With the aid of equations (19)–(21), all the dimensionless quantities can be reduced to dimensional ones and the sought distributions are found:

$$T = T_0 + (1 - S_T(x,y))(T_e - T_0)p \tag{60}$$

$$\mathbf{C} = \mathbf{C}_0 + (\lceil I \rfloor - [S(x,y)])(\Delta\mathbf{C}_0)_T \tag{61}$$

where

$$S_T(x,y) = \operatorname{erf}\left\{\frac{h_0 - y}{4}\sqrt{\left(\frac{Re\,\overline{Pr}}{h_0 x}\right)}\right\};$$

$$[S(x,y)] = [L]\lceil S' \rfloor [L]^{-1},$$

$$S'_i(x,y) = \operatorname{erf}\left\{\frac{h_0 - y}{4}\sqrt{\left(\frac{Re\,\overline{Sc'_i}}{h_0 x}\right)}\right\}$$

$$(\Delta\mathbf{C}_0)_T = \mathbf{V}T_0 + \mathbf{W} - \mathbf{C}_0 + \mathbf{V}(T_e - T_0)p;$$

$$p = \frac{\dfrac{\sqrt{(\bar{\sigma})}}{\hat{\lambda}} \dfrac{\mathbf{r}^{(\mathrm{T})}[D][\bar{E}]^{-0.5}(\mathbf{V}T_0 + \mathbf{W} - \mathbf{C}_0)}{T_e - T_0}}{1 - \dfrac{\sqrt{(\bar{\sigma})}}{\lambda}\mathbf{r}^{(\mathrm{T})}[D][\bar{E}]^{-0.5}\mathbf{V}}. \tag{62}$$

The temperature and concentration distributions (60) and (61) allow one to obtain expressions for the heat and mass fluxes across the interphase. Differentiating equations (60) and (61) at the point $y = h_0$ yields

$$q = -\lambda \left.\frac{\partial T}{\partial y}\right|_{y=h_0} = -\frac{\hat{\lambda}}{2}\sqrt{\left(\frac{u}{\bar{\sigma}x}\right)}(T_e - T_0)p \tag{63}$$

$$\mathbf{N} = -[D]\left.\frac{\partial \mathbf{C}}{\partial y}\right|_{y=h_0} = -\frac{1}{2}\sqrt{\left(\frac{u}{x}\right)}[D][\bar{E}]^{-0.5}(\Delta\mathbf{C}_0)_T. \tag{64}$$

10. CONCLUSION

The method proposed for solving the problem enables one, firstly, with initial parameters being prescribed, to determine the component concentration and temperature profiles in each section of the contacting device and to find their interface, near-wall, integral-mean and other values; secondly, to identify the sought effective length of the apparatus and, thirdly, to theoretically investigate the effect of the parameters of turbulent multicomponent heat and mass transfer on the solution. Furthermore, the method has a number of additional capabilities exemplified by simultaneous absorption of two gases by a nonvolatile turbulent falling film of absorbent.

REFERENCES

1. L. P. Kholphanov, Heat and mass transfer and hydrodynamics of a liquid film flow, *Teor. Osnovy Khim. Tekhnol.* **21**, 86–94 (1987).
2. L. P. Kholpanov and V. Ya. Shkadov, *Hydrodynamics and Heat and Mass Transfer with an Interface*. Izd. Nauka, Moscow (1990).
3. L. P. Kholpanov, V. A. Malyusov and N. M. Zhavoronkov, Simultaneous heat and mass transfer in the systems consisting of a set of droplets and bubbles, *Dokl. AN SSSR* **274,** 890–893 (1984).
4. H. L. Toor, Solution of the linearized equations of multicomponent mass transfer: I, *A.I.Ch.E. Jl* **10,** 448–455 (1964).
5. B. B. Bird, W. E. Stewart and E. N. Lightfoot, *Transport Phenomena*. Wiley, New York (1960).
6. S. R. de Groot and P. Mazur, *Nonequilibrium Thermodynamics*. North-Holland, Amsterdam (1962).
7. W. E. Stewart and R. Prober, Matrix calculation of multicomponent mass transfer in isothermal systems, *Ind. Engng Chem., Fundam.* **3,** 224–235 (1964).
8. V. G. Levich, *Physicochemical Hydrodynamics* (2nd Edn). Fizmatgiz, Moscow (1959).
9. A. V. Luikov, *Heat Conduction Theory*. Izd. Vysshaya Shkola, Moscow (1967).
10. J. O. Hirschfelder, C. F. Curtiss and R. B. Bird, *Molecular Theory of Gases and Liquids*. Wiley, New York (1954).
11. H. L. Toor, C. V. Seshadri and K. D. Arnold, Diffusion and mass transfer in multicomponent mixtures of ideal gases, *A.I.Ch.E. Jl* **11,** 746–747, 755 (1965).
12. G. L. Standart and R. Krishna, A note on multicomponent mass transfer in turbulent flow, *Lett. Heat Mass Transfer* **6,** 35–43 (1979).
13. R. Krishna, A turbulent film model for multicomponent mass transfer, *Chem. Engng J.* **24,** 163–172 (1982).
14. R. Taylor, On multicomponent mass transfer in turbulent flow, *Lett. Heat Mass Transfer* **8,** 397–404 (1981).
15. H. L. Toor, Turbulent diffusion and the multicomponent Reynolds analogy, *A.I.Ch.E. Jl* **6,** 525–527 (1960).
16. L. P. Kholpanov, E. Ya. Kenig, V. A. Malyusov and N. M. Zhavoronkov, Calculation of multicomponent mass transfer in the presence of inert components, *Teor. Osnovy Khim. Tekhnol.* **16,** 729–737 (1982).
17. L. P. Kholpanov, E. Ya. Kenig, V. A. Malyusov and N. M. Zhavoronkov, A new method to calculate of mass transfer two-phase multicomponent systems, *Dokl. AN SSSR* **280,** 684–687 (1985).
18. G. Grossman and M. T. Heath, Simultaneous heat and mass transfer in absorption of gases in turbulent liquid films, *Int. J. Heat Mass Transfer* **27,** 2365–2376 (1984).
19. W. N. Kays, *Convective Heat and Mass Transfer*. McGraw-Hill, New York (1966).
20. A. F. Mills and D. K. Chung, Heat transfer across turbulent falling films, *Int. J. Heat Mass Transfer* **16,** 694–696 (1973).
21. W. H. Henstock and T. J. Hanratty, Gas absorption by a liquid layer flowing on the wall of a pipe, *A.I.Ch.E. Jl* **25,** 122–131 (1979).
22. E. R. van Driest, On turbulent flow near a wall, *J. Aeronaut. Sci.* **23,** 1007–1011, 1036 (1956).
23. H. Reichardt, Vollstandige Darstellung der turbulenten Geschwindigkeitsverteilung in glatten Leitungen, *Z. Angew. Math. Mech.* **31,** 208–219 (1951).
24. A. P. Lamourelle and O. C. Sandall, Gas absorption into a turbulent liquid, *Chem. Engng Sci.* **27,** 1035–1043 (1972).
25. E. Ya. Kenig and L. P. Kholpanov, Two-phase multicomponent mass transfer under conditions of downward phase flow, *J. Engng Phys.* **59,** 99–109 (1990).

26. L. P. Kholpanov, V. N. Babak, V. A. Malyusov and N. M. Zhavoronkov, Study of heat and mass transfer in downward turbulent flow of liquid and gas, *Teor. Osnovy Khim. Tekhnol.* **13,** 323–330 (1979).
27. L. P. Kholpanov, T. B. Babak, V. N. Babak, V. A. Malyusov and N. M. Zhavoronkov, Coupled heat exchange between a liquid film and a solid wall, *J. Engng Phys.* **32,** 1091–1097 (1977).
28. L. P. Kholpanov, E. Ya. Kenig, V. A. Malyusov and V. M. Zhavoronkov, Calculation of two-phase multicomponent mass transfer, *Teor. Osnovy Khim. Tekhnol.* **19,** 579–584 (1985).